Tests for Cointegration, Cobreaking and Cotrending in a System of Trending Variables

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Abstract

We consider a set of variables with a deterministic trend and a stochastic trend. The deterministic trend is allowed to have changes in the intercept and slope. We develop three tests, a cointegration test, a joint test for cointegration and cobreaking, and a joint test for cointegration and cotrending. Our analysis in this paper is complementary to Carrion-i-Silvestre and Kim (2017), which deals with deterministic trends with intecept shifts only.

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1 Introduction

The econometrics literature on unit root and cointegration tests is extremely mature. However, it is relatively recently that variables with both a stochastic and a deterministic trend with breaks draw attention. For the examples of unit root tests, see Carrion-i-Silvestre, Kim, and Perron (2009); Harris, Harvey, Leybourne, and Taylor (2009); Harvey, Leybourne, and Taylor (2012, 2013); and Kim and Perron (2009). Choi (2015) also provides a more extensive review of the unit root literature. For the examples of cointegration tests in the presence of breaks or structural break tests in cointegration models, see Johansen, Mosconi, and Nielsen (2000); Saikkonen and Lütkepohl (2000); Lütkepohl, Saikkonen, and Trenkler (2004); Trenkler, Saikkonen, and Lütkepohl (2007); Harris, Leybourne, and Taylor (2016); Carrion-i-Silvestre and Sansó (2006); Arai and Kurozumi (2007); Qu (2007); Kejriwal and Perron (2010); and Carrion-i-Silvestre and Kim (2017).

This paper is closely related to Carrion-i-Silvestre and Kim (2017). The main idea therein is to view the breaks in the cointegration equation as resulting from the breaks in each variable. This is different from a more traditional view that breaks are exogenously given to cointegration equations and the variables themselves do not have breaks. This kind of difference in modelling the origin of breaks in the cointegration equation might matter less at least asymptotically if breaks are limited to intercepts. However, when slopes are changing, the origin of breaks can even affect the limiting distributions of test statistics.

The analysis in this paper takes the same view as Carrion-i-Silvestre and Kim (2017). However, this paper deals with a model with both intercept shifts and slope changes while Carrion-i-Silvestre and Kim (2017) deals with models with intercept shifts only. The existence of a slope change indeed makes it possible to consistently estimate break dates even in the presence of a stochastic trend, which in turn leads to a testing strategy different from Carrion-i-Silvestre and Kim (2017).

We develop three statistics, namely a robust cointegration test, a joint test for cointegration and cobreaking, and a joint test for cointegration and cotrending. We devise our test statistics as (quasi) log-likelihood ratio tests and derive their limiting distributions. The adequateness of our asymptotics in finite samples is shown via Monte Carlo simulation experiments.

This paper is organized as follows. Section 2 introduces the models and tests and provides the asymptotic results. Section 3 offers Monte Carlo experiment results. Section 4 concludes. The appendix collects some technical derivations.

2 Model and Tests

The observed variables y_t and x_t are assumed to be generated by

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \alpha^{y\prime} \\ \alpha^{x\prime} \end{pmatrix} d_t + \begin{pmatrix} y_t^0 \\ x_t^0 \end{pmatrix}, \tag{1}$$

where y_t is a scalar random variable, x_t is a $p_x \times 1$ random vector, d_t is a $p_d \times 1$ vector of deterministic functions in time, α^y and α^x are coefficient matrices, and y_t^0 and x_t^0 are stochastic components. The deterministic component d_t is specified as

$$d_t' = [DU_t(T_0), \ldots, DU_t(T_m), B_t(T_0), \ldots, B_t(T_m)],$$

where $DU_t(T_j) = 1$ for $t > T_j$ and 0 elsewhere, $B_t(T_j) = (t - T_j)DU_t(T_j)$ and $T_0 = 0$. $DU_t(T_j)$ stands for a shift in the intercept and $B_t(T_j)$ stands for a change in the slope of the linear trend. Models with intercept changes only are analyzed in Carrion-i-Silvestre and Kim (2017). In fact, our strategy to devise test statistics and to derive their limiting distributions extremely resembles theirs.

It is assumed that y_t^0 and x_t^0 are integrated of order one, which means that both y_t and x_t have a stochastic trend in addition to the deterministic trend represented by d_t. We are interested in linear combinations of (y_t, x_t') that cancel out the stochastic trends and/or deterministic trends existing in y_t and x_t. Let $(1, -\beta')$ be such a linear combination. The equation in (1) can be written as

$$y_t = \beta' x_t + \alpha' d_t + v_t, \qquad \text{for } t = 1, \ldots, T, \tag{2}$$

where $\alpha = \alpha^y - \alpha^x \beta$ and $v_t = y_t^0 - \beta' x_t^0$.

For α, we introduce more parameters

$$\alpha' = (\mu_0, \ldots, \mu_m, \psi_0, \ldots, \psi_m),$$

where μ_j is the intercept change at T_j and ψ_j is the slope change. a^y and α^x are similarly defined by (μ_j^y, ψ_j^y) and (μ_j^x, ψ_j^x), respectively. Δ denotes the first difference operator.

Assume that

$$\Delta v_t = \varepsilon_t - \theta \varepsilon_{t-1}, \quad \text{with } v_0 = \varepsilon_0 = 0.$$

Following Carrion-i-Silvestre and Kim (2017), we will use the following definitions throughout the paper:

1. (y_t, x_t') is cointegrated (CI) if and only if $\theta = 1$.

2. (y_t, x_t') is cobreaking (CB) if and only if $\mu_1 = \cdots = \mu_m = \psi_1 = \cdots = \psi_m = 0$.

3. (y_t, x_t') is cotrending (CT) if and only if it is CB and $\psi_0 = 0$.

2.1 Known Break Dates

We first consider testing for the null hypothesis of CI ($\theta = 1$) against the alternative of no CI ($\theta = \bar{\theta} < 1$). We devise our test statistic from the following simple case. Let

$$u_t^x = \Delta x_t^0,$$

and assume

$$(\varepsilon_t, u_t^{x\prime})' \sim IIN\left(0, \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \Sigma_{xx} \end{pmatrix}\right). \tag{3}$$

The normality assumption will be relaxed when the limiting distribution is derived for our test. On the other hand, we deal with only the case in which u_t^x and ε_t are independent. See Carrion-i-Silvestre and Kim (2017) for the case of endogenous regressors.

The model is expressed in matrix notation as

$$y = D\alpha + X\beta + \Psi_\theta^{1/2}\varepsilon, \tag{4}$$

where $y = (y_1, \ldots, y_T)'$, $D = (d_1, \ldots, d_T)'$, $X = (x_1, \ldots, x_T)'$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_T)'$, and

$$\Psi_\theta^{1/2} = \begin{pmatrix} 1 & & & 0 \\ 1-\theta & 1 & & \\ \vdots & \ddots & \ddots & \\ 1-\theta & \cdots & 1-\theta & 1 \end{pmatrix}.$$

Since α and β are irrelevant for CI, they can be concentrated out from the likelihood function. Also, we set $\sigma_\varepsilon^2 = 1$ for simplicity on the exposition, although the final test statistic will be adjusted by a long-run variance estimate. The concentrated log likelihood function is given by

$$L_T(\theta, \pi) = const. - \frac{1}{2} y' \Psi_\theta^{-1/2} M_\theta \Psi_\theta^{-1/2} y, \tag{5}$$

where $\pi = (\pi_1, \ldots, \pi_m)$ is a vector of break fractions where each π_i satisfies $T_i = [\pi_i T]$, $\pi_1 < \cdots < \pi_m$, and

$$M_\theta = I_T - Z^{\bar{\theta}}(Z^{\bar{\theta}\prime} Z^{\bar{\theta}})^{-1} Z^{\bar{\theta}\prime} \text{ with } Z^{\bar{\theta}} = [\Psi_\theta^{-1/2} D, \Psi_\theta^{-1/2} X].$$

When we extend our analysis to the case of unknown break dates, the true break dates and generic break dates must be differentiated. Thus, we will use π^0 and π for the true dates and generic ones.

Our test statistic for CI is the likelihood ratio given by

$$\mathbb{Q}_r \equiv -2\left(L_T(1,\pi^0) - L_T(\bar{\theta},\pi^0)\right). \tag{6}$$

As explained in detail by Carrion-i-Silvestre and Kim (2017), this test is invariant to the CB and CT. In other words, this test provides a way to test for CI irrespective of CB and CT.

Now, we make the following assumptions to derive the asymptotic distribution of our test statistics.

Assumption 1 *$T_i^0 = [\pi_i^0 T]$, and $\pi_i^0 - \pi_j^0 \geq a > 0$ for all $0 \leq j < i \leq m+1$ where $\pi_0^0 = 0$ and $\pi_{m+1}^0 = 1$.*

Assumption 2 *$\varepsilon_t = \sum_{i=0}^{\infty} c_i \eta_{t-i}$ with $\sum_{i=0}^{\infty} i \|c_i\| < \mathcal{C}$ and $\eta_t \sim i.i.d.(0,1)$, $u_t^x = \sum_{i=0}^{\infty} G_i \eta_{t-i}^x$ with $\sum_{i=0}^{\infty} i \|G_i\| < \mathcal{C}$, $G_\infty = \sum_{i=0}^{\infty} G_i$ is of full column rank, and $\eta_t^x \sim i.i.d.(0, I_{p_x})$, and η_t and η_t^x are independent.*

Assumption 3 *ψ^x is of full column rank so that $(\psi^{x\prime}\psi^x)^{-1}$ exists.*

These assumptions are identical to those in Carrion-i-Silvestre and Kim (2017). The next theorem states the asymptotic distribution of the $\mathbb{Q}_r$ statistic.

Theorem 1 *Let $\theta = 1 - \lambda/T$, $\bar{\theta} = 1 - \bar{\lambda}/T$ and $\omega_\varepsilon = \sum_{i=0}^{\infty} c_i$. Then, under Assumptions 1 $\sim$ 2, we have*

$$\begin{aligned}
\frac{1}{\omega_\varepsilon^2}\mathbb{Q}_r &\Rightarrow \varphi_r(\lambda,\bar{\lambda},\pi^0) \\
&= \Phi_1(\lambda,\bar{\lambda}) - \Phi_2(\lambda,0,\pi^0;m+1,m+1,p_x) + \Phi_2(\lambda,\bar{\lambda},\pi^0;m+1,m+1,p_x),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_1(\lambda,\bar{\lambda}) &= 2\bar{\lambda}\int_0^1 V_{\bar{\lambda}}^{\lambda}(s)\, dV^{\lambda}(s) - \bar{\lambda}^2 \int_0^1 V_{\bar{\lambda}}^{\lambda}(s)^2\, ds, \\
\Phi_2(\lambda,\bar{\lambda},\pi;p,q,n) &= \left(\int_0^1 Q_{\bar{\lambda}}^{\pi}(s)\, dV_{\bar{\lambda}}^{\lambda}(s)\right)' \left(\int_0^1 Q_{\bar{\lambda}}^{\pi}(s)\, Q_{\bar{\lambda}}^{\pi\prime}(s)\, ds\right)^{-1} \left(\int_0^1 Q_{\bar{\lambda}}^{\pi}(s)\, dV_{\bar{\lambda}}^{\lambda}(s)\right), \\
V_{\bar{\lambda}}^{\lambda}(s) &= V^{\lambda}(s) - \bar{\lambda}\int_0^s e^{-\bar{\lambda}(s-r)} V^{\lambda}(r) dr, \\
V^{\lambda}(s) &= V(s) + \lambda \int_0^s V(r) dr, \\
Q_{\bar{\lambda}}^{\pi}(s) &= Q^{\pi}(s|p,q,n) - \bar{\lambda}\int_0^s e^{-\bar{\lambda}(s-r)} Q^{\pi}(r|p,q,n) dr, \\
Q^{\pi}(s|p,q,n) &= (du(s,\pi_0),\ldots,du(s,\pi_{p-1}), b(s,\pi_0),\ldots,b(s,\pi_{q-1}), W_n(s)')', \\
du(r,\pi_j) &= 1(r > \pi_j), \\
b(r,\pi_j) &= (r-\pi_j)1(r > \pi_j),
\end{aligned}$$

$V(r)$ and $W(r)$ are independent standard Wiener processes of dimensions 1 and p_x respectively, and $W_n(r)$ collects the first n elements of $W(r)$.

Note that $\varphi_r(\lambda, \bar{\lambda}, \pi^0)$ stands for the null distribution when $\lambda = 0$ and stands for the alternative distribution when $\lambda > 0$. In practice, the value of $\bar{\theta}$ has to be decided by the researcher. Following the custom in the econometrics literature, we recommend to set the value of $\bar{\theta}$ so that the local asymptotic power curve is tangent to the theoretical power envelop with 50% power. Table 1 reports[1] the suggested values for $\bar{\theta}(= 1 - \bar{\lambda}/T)$. With this choice of $\bar{\lambda}$, selected percentiles of $\varphi(\lambda, \bar{\lambda}, \pi^0)$ are reported in Table 2.[2]

Now, we develop joint tests for CI and CB and for CI and CT. Now that α is of interest, we work with the unconcentrated log likelihood $L_T(\alpha, \beta|\theta, \pi^0)$. Let $\mathbb{Q}_{cb}$ be the test statistic for the joint null of CI and CB and $\mathbb{Q}_{ct}$ be the joint null of CI and CT. Then, we propose

$$\begin{aligned} \mathbb{Q}_{cb} &= -2\left(\max_{\substack{\alpha,\beta \\ s.t. \\ R_{cb}\alpha=0}} L_T(\alpha, \beta|1, \pi^0) - \max_{\alpha,\beta} L_T(\alpha, \beta|\bar{\theta}, \pi^0) \right), \\ \mathbb{Q}_{ct} &= -2\left(\max_{\substack{\alpha,\beta \\ s.t. \\ R_{ct}\alpha=0}} L_T(\alpha, \beta|1, \pi^0) - \max_{\alpha,\beta} L_T(\alpha, \beta|\bar{\theta}, \pi^0) \right), \end{aligned}$$

where

$$\begin{aligned} R_{cb}\alpha &= (\mu_1 \ldots, \mu_m, \psi_1, \ldots \psi_m)', \\ R_{ct}\alpha &= (\mu_1 \ldots, \mu_m, \psi_0, \psi_1, \ldots \psi_m)'. \end{aligned}$$

The asymptotic null distributions for the $\mathbb{Q}_{cb}$ and $\mathbb{Q}_{ct}$ statistics are reported in the following theorem.

Theorem 2 *Let $\theta = 1$, $\bar{\theta} = 1 - \bar{\lambda}/T$ and $\omega_\varepsilon = \sum_{i=0}^{\infty} c_i$. Suppose Assumptions 1 $\sim$ 3 hold. Let $\Phi_1(\lambda, \bar{\lambda})$ and $\Phi_2(\lambda, \bar{\lambda}, \pi; p, q, n)$ be the same as in Theorem 1.*

(i) If CB holds and $p_x \geq m$, we have

$$\begin{aligned} \frac{1}{\omega_\varepsilon^2}\mathbb{Q}_{cb} &\Rightarrow \varphi_{cb}(0, \bar{\lambda}, \pi^0) \\ &\equiv \Phi_1(0, \bar{\lambda}) - \Phi_2(0, 0, \pi^0; 1, m+1, p_x - m) + \Phi_2(0, \bar{\lambda}, \pi^0; m+1, m+1, p_x). \end{aligned}$$

[1] In Tables 1 and 2, we report the results for the case of $p_x = 1$ and 2. The extended tables for $p_x = 3, \ldots, 5$ are available upon request from the authors.

[2] To generate these critical values, a Wiener process is approximated with 2,000 steps and the number of simulation repetitions is roughly 20,000.

(ii) If CT holds and $p_x \geq m+1$, we have

$$\begin{aligned}\frac{1}{\omega_\varepsilon^2}\mathbb{Q}_{ct} &\Rightarrow \varphi_{ct}(0,\bar{\lambda},\pi^0) \\ &\equiv \Phi_1(0,\bar{\lambda}) - \Phi_2(0,0,\pi^0;1,m+1,p_x-m-1) + \Phi_2(0,\bar{\lambda},\pi^0;m+1,m+1,p_x).\end{aligned}$$

It is worth mentioning that when there are more breaks than stochastic regressors ($p_x \geq m$ for CB and $p_x \geq m+1$ for CT), the limiting distributions depend on nuisance parameters, and thus cannot be tabulated. Relevant percentiles of $\varphi_{cb}(0,\bar{\lambda},\pi^0)$ can be found in Tables 3 and 4.

2.2 Unknown Break Dates

Now we consider using estimated break dates instead of the true ones. When the slopes are changing, the break dates can be estimated regardless of CI. More specifically, the break dates can be estimated by minimizing the sum of squared residuals from a regression of $q_t = (\Delta y_t, \Delta x_t')'$ on Δd_t where d_t is defined with a generic π. Let $q = [q_2, \ldots, q_T]'$ and $\Delta D_\pi = [\Delta d_2, \ldots, \Delta d_T]'$. Then the break date estimator is defined to be

$$\begin{aligned}\widetilde{\pi} &= (\widetilde{\pi}_1, \ldots, \widetilde{\pi}_m) \\ &= \arg\min_{\pi} tr\left[q'(I - \Delta D_\pi(\Delta D_\pi'\Delta D_\pi)^{-1}\Delta D_\pi')q\right].\end{aligned}$$

The $\mathbb{Q}_r$, $\mathbb{Q}_{cb}$ and $\mathbb{Q}_{ct}$ tests can be constructed at the estimated break dates, $\widetilde{\pi}$. Under Assumption 3, $\widetilde{\pi}$ is known to be consistent at rate T.

Now, our test statistics are given by

$$\begin{aligned}\widetilde{\mathbb{Q}}_r &= -2\left(\max_{\alpha,\beta} L_T(\alpha,\beta|1,\widetilde{\pi}) - \max_{\alpha,\beta} L_T(\alpha,\beta|\bar{\theta},\widetilde{\pi})\right) \\ \widetilde{\mathbb{Q}}_{cb} &= -2\left(\max_{\substack{\alpha,\beta \\ s.t. \\ R_{cb}\alpha=0}} L_T(\alpha,\beta|1,\pi) - \max_{\alpha,\beta} L_T(\alpha,\beta|\bar{\theta},\widetilde{\pi})\right) \\ \widetilde{\mathbb{Q}}_{ct} &= -2\left(\max_{\substack{\alpha,\beta \\ s.t. \\ R_{ct}\alpha=0}} L_T(\alpha,\beta|1,\pi) - \max_{\alpha,\beta} L_T(\alpha,\beta|\bar{\theta},\widetilde{\pi})\right).\end{aligned}$$

The next theorem states that these test statistics have the asymptotic distributions that have been derived for the known break date case.

Theorem 3 *Let $\theta = 1$, $\bar{\theta} = 1 - \bar{\lambda}/T$ and $\omega_\varepsilon = \sum_{i=0}^\infty c_i$. Suppose that Assumptions 1 ~ 3 hold and $||\widetilde{\pi} - \pi^0|| = O_p(T^{-1})$.*

(i)

$$\frac{1}{\omega_\varepsilon^2}\widetilde{\mathbb{Q}}_r \Rightarrow \varphi(0,\bar{\lambda},\pi^0),$$

where $\varphi(0, \bar{\lambda}, \pi^0)$ is as defined in Theorem 1.

(ii) If CB holds and $p_x \geq m$,

$$\frac{1}{\omega_\varepsilon^2} \widetilde{\mathbb{Q}}_{cb} \Rightarrow \varphi_{cb}(0, \bar{\lambda}, \pi^0),$$

where $\varphi_{cb}(0, \bar{\lambda}, \pi^0)$ is as defined in Theorem 2.

(iii) If CT holds and $p_x \geq m + 1$,

$$\frac{1}{\omega_\varepsilon^2} \widetilde{\mathbb{Q}}_{ct} \Rightarrow \varphi_{ct}(0, \bar{\lambda}, \pi^0),$$

where $\varphi_{ct}(0, \bar{\lambda}, \pi^0)$ is as defined in Theorem 2.

3 Simulation

The data generation process (DGP) is identical to (1). Much of the simulation design resembles the one employed in Carrion-i-Silvestre and Kim (2017) to make the comparison easier. We let α^x be a matrix of 2, and β be a $p_x \times 1$ vector of $1/p_x$s. α^y is determined in such a way that α can have a desired value from the relationship $\alpha = \alpha^y - \alpha^x \beta$. $\mu_0 = 0$ in all cases as this does not affect the results. x_t^0 is an accumulation of independent standard normal errors. y_t^0 is a sum of $\beta' x_t^0$ and $v_t = \sum_{j=1}^t (\varepsilon_j - \theta \varepsilon_{j-1})$ with ε_t being an independent standard normal variate and $v_0 = \varepsilon_0 = 0$.

The following 6 DGPs are simulated with $p_x = 2$. The number of simulation repetitions is 1,000 in all cases.

DGP 1 (CI + CT) $\theta = 1$, $\mu_i = \psi_j = 0$, $\psi_0 = 0$.

DGP 2 (CI + CB) $\theta = 1$, $\mu_i = \psi_j = 0$, $\psi_0 = 0.3$.

DGP 3 (CI + No CB) $\theta = 1$, $\mu_i = 10\psi_j = 0.5 \sim 2.5$ in steps of 0.5 and $3 \sim 12$ in steps of 3, $\psi_0 = 0.3$.

DGP 4 (No CI + CT) $\theta = 0$, 0.25, 0.5 and $1.05 \sim 1.25$ in steps of 0.05, $\mu_i = \psi_j = 0$, $\psi_0 = 0$.

DGP 5 (No CI + CB) $\theta = 0$, 0.25, 0.5 and $1.05 \sim 1.25$ in steps of 0.05, $\mu_i = \psi_j = 0$, $\psi_0 = 0.3$.

DGP 6 (No CI + No CB) $\theta = 1.05 \sim 1.25$ in steps of 0.05 with $\mu_i = 10\psi_j = 10(\theta - 1)$, and $\theta = 0$ with $\mu_i = 10\psi_j = 3 \sim 12$ in steps of 3, $\psi_0 = 0.3$.

For all tests, the long-run variance is estimated by a heteroskedasticity and autocorrelation consistent covariance estimator with the quadratic spectral kernel, for which the bandwidth parameter is selected using Andrews's (1991) data dependent method with an AR(1) approximation.

Finite Sample Size Table 5 reports the finite sample sizes for nominal 5% significance. In the simulation, the break dates are estimated by minimizing the sum of squared residuals from a regression of first differences of (y_t, x_t') on intercept shifts and impulse dummies.

Overall, the actual sizes are quite close to the nominal level, which suggests that the asymptotic critical values offer a good approximation.

Power Comparison Table 6 reports size adjusted powers. We focus on the tests using estimated break dates. For comparison, also simulated are the Gregory and Hansen (1996) test (GH), the Shin (1994) test, and the Kejriwal and Perron (2010) test (KP).

The finite sample critical values are obtained under DGP1 for all tests but the GH test. The critical value for the GH test is obtained under DGP4 because this test takes the null of no CI. As a result, the rejection probabilities are 5% exactly for all tests but the GH test. The GH test rejects the null with 100%, which is exactly as it is intended.

With DGP 2, CI and CB still hold but CT does not. Hence, the $\widetilde{\mathbb{Q}}_{ct}(1)$ test correctly detects the breakdown of CT, while all other tests show the same rejection probabilities as in DGP1.

With DGP 3, both $\widetilde{\mathbb{Q}}_{cb}(1)$ and $\widetilde{\mathbb{Q}}_{ct}(1)$ reject the joint null with large probabilities since CB does not hold. $\widetilde{\mathbb{Q}}_{ct}(1)$ appears more powerful than $\widetilde{\mathbb{Q}}_{cb}(1)$. On the other hand, $\widetilde{\mathbb{Q}}_{r}(1)$ still remains at 5% because CI still holds. This difference in rejection probabilities shows that our tests are working exactly in the way they are designed. Note that our tests are more powerful than the Shin test and the KP test. Also noteworthy is the fact that the power of the Shin test is decreasing as μ_1 gets larger, which is the well-documented non-monotonic power problem.

With DGP 4, not only the $\widetilde{\mathbb{Q}}_{cb}(1)$ and $\widetilde{\mathbb{Q}}_{ct}(1)$ tests but also the $\widetilde{\mathbb{Q}}_{r}(1)$ test rejects with large probabilities because CI does not hold. Again note that our tests are performing better than the Shin test and the KP test. The GH test is supposed to control the size, but it apparently fails.

With DGP 5, the rejection rates for $\widetilde{\mathbb{Q}}_{ct}(1)$ become very close to one due to the break down of CT. With DGP 6, the results are similar to the ones for DGP 5.

4 Conclusion

We consider a system of trending variables and develop three statistics, a CI test, a joint test for CI and CB, and a joint test for CI and CT. Our analysis in this paper is complementary to Carrion-i-Silvestre and Kim (2017), with the notable difference that we allow for slope changes. When slopes are changing, the break dates are estimated consistently regardless of CI. This feature enables us to implement a test procedure that is different from Carrion-i-Silvestre and Kim (2017).

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<Appendix>

Proof of Theorem 1: We only need to find Υ so that $\Upsilon Z_{[Tr]}$ converges properly. Let

$$\Upsilon = \begin{pmatrix} \Upsilon_1 & 0 \\ -T^{-1/2}G_\infty^{-1/2}\alpha^{x\prime} & T^{-1/2}G_\infty^{-1/2} \end{pmatrix},$$

where $\Upsilon_1 = diag\{I_{m+1}, T^{-1}I_{m+1}\}$. It follows that

$$\Upsilon Z_{[Tr]} = \begin{pmatrix} DU_{[Tr]}(T_0) \\ \vdots \\ DU_{[Tr]}(T_m) \\ T^{-1/2}G_\infty^{-1/2}x_{[Tr]} - T^{-1/2}G_\infty^{-1/2}\alpha^{x\prime}d_{[Tr]} \end{pmatrix} = \begin{pmatrix} DU_{[Tr]}(T_0) \\ \vdots \\ DU_{[Tr]}(T_m) \\ T^{-1/2}G_\infty^{-1/2}x^0_{[Tr]} \end{pmatrix} \Rightarrow Q^\pi(r).$$

Then, the rest of the proof is the same as the proof of Theorem 1 in Carrion-i-Silvestre and Kim (2017). □

Proof of Theorem 2: See the proof for Theorem 2 in Carrion-i-Silvestre and Kim (2017). □

Proof of Theorem 3: The result follows from a similar argument to the proof of Theorem 4(i) in Carrion-i-Silvestre and Kim (2017). Because we have the $B_t(T_j)$ terms in addition to the $DU_t(T_j)$ terms, let $\widetilde{D} = [\widetilde{d}_1, \ldots, \widetilde{d}_T]'$. $\widetilde{d}_t$ is the same as d_t except that $DU_t(T_j^0)$ is replaced by $DU_t(\widetilde{T}_j)$ and $B_t(T_j^0)$ by $B_t(\widetilde{T}_j)+(\widetilde{T}_j - T_j^0)DU_t(T_j)$ for $j = 1, \ldots, m$, where $(\widetilde{T}_1, \ldots, \widetilde{T}_m) = T\widetilde{\pi} = T(\widetilde{\pi}_1, \ldots, \widetilde{\pi}_m)$.

Then, the columns of $D - \widetilde{D}$ are $DU(T_j^0) - DU(\widetilde{T}_j)$ and $B_t(T_j^0) - B(\widetilde{T}_j) - (\widetilde{T}_j - T_j^0)DU(\widetilde{T}_j)$ for $j = 0, \ldots, m$, and each of these columns has non-zero elements only between $\min(T_j^0, \widetilde{T}_j) + 1$ and $\max(T_j^0, \widetilde{T}_j)$, which are bounded. The rest of the proof is analogous to the proof of Theorem 4(i) in Carrion-i-Silvestre and Kim (2017) with the simplification that the same break date estimates are used for both the null and alternative regressions. □

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Table 1. Suggested Values of $\bar{\lambda}$, Local to Unity Parameter

m	0	1	2	3
$p_x = 1$	13.4	18.0	22.6	27.7
$p_x = 2$	14.9	19.4	23.7	28.9

Table 2. Upper Percentiles for $\varphi_r(0, \bar{\lambda}, \pi^0)$

(i) One Break, $m = 1$

	$p_x = 1$				$p_x = 2$		
π^0	10%	5%	1%	π^0	10%	5%	1%
0.2	−9.08	−7.92	−5.35	0.2	−10.24	−9.11	−6.78
0.3	−9.40	−8.36	−6.11	0.3	−10.38	−9.31	−6.76
0.4	−9.49	−8.49	−6.30	0.4	−10.48	−9.46	−7.14
0.5	−9.52	−8.48	−6.38	0.5	−10.51	−9.45	−7.30
0.6	−9.52	−8.49	−6.12	0.6	−10.46	−9.46	−6.99
0.7	−9.38	−8.31	−6.03	0.7	−10.43	−9.29	−6.95
0.8	−9.11	−7.91	−5.56	0.8	−10.16	−9.03	−6.67

(ii) Two Breaks, $m = 2$

	$p_x = 1$				$p_x = 2$		
π^0	10%	5%	1%	π^0	10%	5%	1%
(0.2, 0.4)	−12.54	−11.49	−9.01	(0.2, 0.4)	−13.54	−12.49	−10.05
(0.2, 0.5)	−12.81	−11.73	−9.44	(0.2, 0.5)	−13.68	−12.61	−10.23
(0.2, 0.6)	−12.87	−11.78	−9.55	(0.2, 0.6)	−13.65	−12.63	−10.24
(0.2, 0.7)	−12.77	−11.67	−9.26	(0.2, 0.7)	−13.69	−12.62	−10.18
(0.2, 0.8)	−12.59	−11.42	−8.94	(0.2, 0.8)	−13.43	−12.36	−10.03
(0.3, 0.5)	−12.75	−11.66	−9.28	(0.3, 0.5)	−13.66	−12.55	−10.30
(0.3, 0.6)	−12.96	−11.98	−9.79	(0.3, 0.6)	−13.77	−12.77	−10.48
(0.3, 0.7)	−13.00	−12.01	−9.87	(0.3, 0.7)	−13.80	−12.72	−10.57
(0.3, 0.8)	−12.83	−11.72	−9.29	(0.3, 0.8)	−13.63	−12.52	−10.24
(0.4, 0.6)	−12.81	−11.71	−9.46	(0.4, 0.6)	−13.71	−12.69	−10.48
(0.4, 0.7)	−13.00	−12.03	−9.88	(0.4, 0.7)	−13.77	−12.80	−10.64
(0.4, 0.8)	−12.89	−11.79	−9.45	(0.4, 0.8)	−13.62	−12.58	−10.37
(0.5, 0.7)	−12.79	−11.70	−9.46	(0.5, 0.7)	−13.68	−12.59	−10.22
(0.5, 0.8)	−12.74	−11.66	−9.26	(0.5, 0.8)	−13.62	−12.56	−10.30
(0.6, 0.8)	−12.58	−11.44	−9.03	(0.6, 0.8)	−13.50	−12.39	−10.02

(iii) Three Breaks, $m = 3$

	$p_x = 1$				$p_x = 2$		
π^0	10%	5%	1%	π^0	10%	5%	1%
(0.2, 0.4, 0.6)	−16.43	−15.33	−12.95	(0.2, 0.4, 0.6)	−17.37	−16.27	−13.92
(0.2, 0.4, 0.7)	−16.54	−15.47	−13.22	(0.2, 0.4, 0.7)	−17.41	−16.33	−14.20
(0.2, 0.4, 0.8)	−16.44	−15.35	−13.16	(0.2, 0.4, 0.8)	−17.28	−16.23	−13.63
(0.2, 0.5, 0.7)	−16.51	−15.44	−13.10	(0.2, 0.5, 0.7)	−17.37	−16.30	−14.01
(0.2, 0.5, 0.8)	−16.47	−15.40	−13.07	(0.2, 0.5, 0.8)	−17.38	−16.33	−13.98
(0.2, 0.6, 0.8)	−16.39	−15.28	−12.95	(0.2, 0.6, 0.8)	−17.29	−16.17	−13.83
(0.3, 0.5, 0.7)	−16.57	−15.55	−13.34	(0.3, 0.5, 0.7)	−17.37	−16.25	−14.07
(0.3, 0.5, 0.8)	−16.55	−15.48	−13.26	(0.3, 0.5, 0.8)	−17.40	−16.31	−13.95
(0.3, 0.6, 0.8)	−16.59	−15.53	−13.19	(0.3, 0.6, 0.8)	−17.40	−16.36	−14.00
(0.4, 0.6, 0.8)	−16.43	−15.34	−12.89	(0.4, 0.6, 0.8)	−17.31	−16.22	−13.81

Table 3. Upper Percentiles for $\varphi_{cb}(0, \bar{\lambda}, \pi^0)$

	$p_x = 1$				$p_x = 2$		
π^0	10%	5%	1%	π^0	10%	5%	1%
0.2	−6.04	−4.47	−0.85	0.2	−7.06	−5.44	−1.83
0.3	−6.36	−4.83	−1.52	0.3	−7.22	−5.67	−2.40
0.4	−6.34	−4.85	−1.39	0.4	−7.40	−5.85	−2.38
0.5	−6.34	−4.85	−1.30	0.5	−7.39	−5.89	−2.38
0.6	−6.46	−4.95	−1.45	0.6	−7.43	−6.02	−2.67
0.7	−6.31	−4.72	−1.39	0.7	−7.31	−5.89	−2.29
0.8	−6.14	−4.50	−1.03	0.8	−7.13	−5.50	1.89

Table 4. Upper Percentiles for $\varphi_{ct}(0, \bar{\lambda}, \pi^0)$

	$p_x = 2$		
π^0	10%	5%	1%
0.2	−5.66	−3.94	−0.24
0.3	−5.90	−4.25	−0.44
0.4	−5.83	−4.17	−0.47
0.5	−5.88	−4.11	−0.35
0.6	−5.94	−4.29	−0.55
0.7	−5.84	−4.14	−0.47
0.8	−5.77	−3.95	−0.39

Table 5. Finite Sample Sizes, DGP 1 (CI+CT)

π^0	T	$\widetilde{\mathbb{Q}}_r(m)$	$\widetilde{\mathbb{Q}}_{cb}(m)$	$\widetilde{\mathbb{Q}}_{ct}(m)$	$\mathbb{Q}_r(m)$	$\mathbb{Q}_{cb}(m)$	$\mathbb{Q}_{ct}(m)$
(0.5)	120	.049	.058	.065	.048	.059	.067
	360	.055	.058	.049	.056	.048	.050
(0.3, 0.7)	120	.043	.061	–	.043	.053	–
	360	.057	.055	–	.058	.055	–
(0.2, 0.5, 0.8)	120	.035	–	–	.032	–	–
	360	.042	–	–	.044	–	–

Table 6. Power Comparison, Null Rejection Probabilities

	θ	μ_1	ψ_1	$\widetilde{\mathbb{Q}}_r(1)$	$\widetilde{\mathbb{Q}}_{cb}(1)$	$\widetilde{\mathbb{Q}}_{ct}(1)$	Shin	KP	GH
DGP 1	1.0	0.0	0.0	.05	.05	.05	.05	.05	1.0
DGP 2	1.0	0.0	0.3	.05	.05	1.0	.05	.05	1.0
DGP 3	1.0	0.5	0.3	.05	.13	1.0	.12	.10	1.0
		1.0		.05	.40	1.0	.35	.26	1.0
		1.5		.05	.73	1.0	.57	.55	1.0
		2.0		.05	.92	1.0	.73	.81	1.0
		2.5		.05	.99	1.0	.83	.94	1.0
		3.0		.05	1.0	1.0	.88	.99	1.0
		6.0		.05	1.0	1.0	.85	1.0	1.0
		9.0		.05	1.0	1.0	.64	1.0	1.0
		12.0		.05	1.0	1.0	.41	1.0	1.0
DGP 4	0.0	0.0	0.0	1.0	1.0	1.0	.42	.78	.05
	0.25			1.0	1.0	1.0	.21	.72	.48
	0.50			1.0	1.0	1.0	.40	.78	.99
	1.05			.08	.09	.13	.09	.07	1.0
	1.10			.17	.20	.39	.19	.16	1.0
	1.15			.32	.38	.66	.35	.29	1.0
	1.20			.48	.56	.80	.48	.43	1.0
	1.25			.63	.71	.88	.56	.56	1.0
DGP 5	0.0	0.0	0.3	1.0	1.0	1.0	.42	.78	.05
	0.25			1.0	1.0	1.0	.21	.72	.48
	0.50			1.0	1.0	1.0	.40	.78	.99
	1.05			.08	.09	1.0	.09	.07	1.0
	1.10			.17	.20	1.0	.19	.16	1.0
	1.15			.32	.38	1.0	.35	.29	1.0
	1.20			.48	.56	1.0	.48	.43	1.0
	1.25			.63	.71	1.0	.56	.56	1.0
DGP 6	1.05	0.5	0.3	.08	.17	1.0	.16	.12	1.0
	1.10	1.0		.17	.50	1.0	.41	.36	1.0
	1.15	1.5		.32	.77	1.0	.58	.59	1.0
	1.20	2.0		.49	.92	1.0	.68	.72	1.0
	1.25	2.5		.63	.97	1.0	.72	.81	1.0
	0.0	3.0		1.0	1.0	1.0	.44	.78	.04
		6.0		1.0	1.0	1.0	.43	.81	.12
		9.0		1.0	1.0	1.0	.45	.86	.32
		12.0		1.0	1.0	1.0	.48	.92	.59
		15.0		1.0	1.0	1.0	.50	.97	.80